14, 3, 1985.
15. BLAGODATSKIKH V.I. and FILIPPOV A.F., Differential inclusions and optimal control. Trudy Mat. Inst. Akad. Nauk SSSR, 169, 1985.
16. BAIBAZAROV M. and SUBBOTIN A.I., On a definition of the value of a differential game. Differents. Uravn., 20, 2, 1984.

Translated by D.L.

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CONDITIONS FOR A SUM OF FORMS TO BE OF FIXED SIGN AND FOR STABILITY OF MOTION ON MANIFOLDS*

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Lyapunov's corollary of the Stability Theorem /1/, a special case of which is Routh's theorem on the stability of the steady motion of a system with cyclic coordinates, provides a point of departure for the investigation conducted in this paper of the stability of motion on manifolds, particularly those defined by the integrals of the equations of the perturbed motion. Sufficient conditions are obtained for a sum of forms to be positive- or negative-definite and for the motion of polynomial systems to be stable on these manifolds.

1. Given a sum of forms

$$F(\mathbf{x}) = \sum_{s=2}^{2q} X^{(s)}(\mathbf{x}, A_{i_1\ldots i_s}), \quad \mathbf{x} = (x_1, \ldots, x_n) \in R_x^n \tag{1.1}$$

and a manifold M defined by equalities

$$F_r(\mathbf{x}) = \sum_{s=1}^{p} X_r^{(s)}(\mathbf{x}, B_{ri_1\ldots i_s}) = 0, \quad r = 1, 2, \ldots, m; m < n, p < q \tag{1.2}$$

where $X^{(s)}(\mathbf{x}, A_{i_1\ldots i_s})$, $X_r^{(s)}(\mathbf{x}, B_{ri_1\ldots i_s})$ are multilinear forms of degree s, of the form

$$X^{(s)}(x, A_{i_1\ldots i_s}) = \sum_{i_1=1}^{n} \cdots \sum_{i_s=i_{s-1}}^{n} A_{i_1\ldots i_s} x_{i_1} \ldots x_{i_s}$$

$A_{i_1,\ldots,i_s}$, $B_{ri_1,\ldots,i_s}$ are real numbers, p, q, s, m, n are positive integers, and R_x^n is Euclidean n-space. Like terms in the forms are reduced and the terms are assumed to be lexicographically ordered.

We shall determine the sufficient conditions for functions (1.1) to be positive- or negative-definite under constraints (1.2).

Let R^N_y denote the Euclidean space of vectors $\mathbf{y} = (y_1, \ldots, y_N)$ and $\Phi : R_x^n \to R_y^N$ the mapping defined as follows:

$$\begin{aligned} &y_1 = x_1^q, \quad y_2 = x_1^{q-1}x_2, \quad \ldots, \quad y_n = x_1^{q-1}x_n \\ &y_{n+1} = x_1^{q-2}x_2^2, \quad y_{n+2} = x_1^{q-2}x_2x_3, \ldots, y_{N-n+1} = x_1, \\ &y_{N-n+2} = x_2, \ldots, y_N = x_n \end{aligned} \tag{1.3}$$

i.e., $y_j = x_{i_1}x_{i_2}\ldots x_{i_s}$ and $j \rightleftarrows i_1 i_2 \ldots i_s$, where $i_1 \leqslant i_2 \leqslant \ldots \leqslant i_s$, $i_1, i_2, \ldots, i_s = 1, 2, \ldots, n$; $j = 1, 2, \ldots, N$.

Lemma 1.1. A sum of forms $F(\mathbf{x})$ (1.1) defined in R_x^n is mapped by Φ (1.3) into a certain quadratic form (q.f.)

$$f(\mathbf{y}) = \sum_{j_1, j_2}^{N} c_{j_1 j_2} y_{j_1} y_{j_2}, \quad c_{j_1 j_2} = c_{j_2 j_1} \tag{1.4}$$

defined in R_y^N.

Proof. Let us assume that the forms in the sum $F(\mathbf{x})$ follow each other in decreasing order of their degrees, beginning with a form of degree $2q$ and ending with a second-degree form. We shall show that each term in $F(\mathbf{x})$ can be mapped by a mapping Φ of type (1.3) into some monomial of a second-degree q.f. $f(\mathbf{y})$.

We begin with a second-degree term in the function $F(\mathbf{x})$. Using the equalities

$$x_1 = y_{N-n+1},\ x_2 = y_{N-n+2}, \ldots, x_n = y_N$$

appearing in the mapping (1.3), we see that a q.f. in the sum of forms $F(\mathbf{x})$ is mapped into a unique q.f. in the variables $y_{N-n+1}, y_{N-n+2}, \ldots, y_N$.

Consider a third-degree form in $F(\mathbf{x})$. Together with the variables $y_{N-n+1}, y_{N-n+2}, \ldots, y_N$, we use the variables

$$y_j = x_{i_1} x_{i_2}\ (i_1 \leqslant i_2;\ i_1, i_2 = 1, 2, \ldots, n)$$

When this is done each third-order from in $F(\mathbf{x})$ is mapped into a second-degree term in the q.f. $f(\mathbf{y})$. For example, the expression $x_{n-1}x_n^2$ is mapped into a product of coordinates $y_{N-1}y_{N-n}$. But this mapping is not unique, since $x_{n-1}x_n^2 = x_{n-1}x_n x_n$ also goes into the product $y_{N-n-1}y_N$.

Note that third-order terms in $F(\mathbf{x})$ may also be mapped into first-order terms, if one uses the components of Φ of the form $y_j = x_{i_1}x_{i_2}x_{i_3}$ $(i_1 \leqslant i_2 \leqslant i_3,\ i_1, i_2, i_3 = 1, 2, \ldots, n)$. However, only transformations into a quadratic form interest us here.

Now consider an arbitrary term of order s in $F(\mathbf{x})$:

$$A_{i_1 i_2 \ldots i_s} x_{i_1} x_{i_2} \cdots x_{i_s}\ (i_1 \leqslant i_2 \leqslant \cdots \leqslant i_s;\ i_1, i_2, \ldots, i_s = 1, \ldots, n,$$
$$s = 2q,\ 2q-1, \ldots, 2)$$

We divide the multi-index $i_1 i_2 \ldots i_s$ into two parts: $i_1 i_2 \ldots i_k$ and $i_{k+1} i_{k+2} \ldots i_s$ $(1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant n,\ 1 \leqslant i_{k+1} \leqslant \cdots \leqslant i_s \leqslant n)$, in such a way that Φ preserves the one-to-one correspondence

$$i_1 i_2 \ldots i_k \rightleftarrows j_1, \quad i_{k+1} i_{k+2} \ldots i_s \rightleftarrows j_2$$

If one takes $k \in \{1, 2, \ldots, q\}$, $(s-k) \in \{1, 2, \ldots, q\}$, such a partition is always possible, since $s \in \{2, 3, \ldots, 2q\}$. Then any s-th-order term in $F(\mathbf{x})$ is mapped into a corresponding term $c_{j_1 j_2} y_{j_1} y_{j_2}$ in the q.f. $f(\mathbf{y})$, and hence the sum of forms $F(\mathbf{x})$ is mapped into the q.f. $f(\mathbf{y})$.

As a rule, a transformation of the type described is not unique.

In order to determine the dependence between the coefficients $c_{j_1 j_2}$ and the coefficients $A_{i_1 \ldots i_s}$ of $F(\mathbf{x})$, we equate $f(y)$ and $F(\mathbf{x})$:

$$\sum_{j_1 j_2}^{N} c_{j_1 j_2} y_{j_1} y_{j_2} = \sum_{s=2}^{2q} X^{(s)}(\mathbf{x}, A_{i_1 \ldots i_s}) \tag{1.5}$$

and substitute throughout the values of $y_1, \ldots, y_N$ from (1.3). Equating coefficients of like terms on the left and right of (1.5), we obtain a system of linear relationships between $c_{j_1 j_2}$ and $A_{i_1 \ldots i_s}$:

$$c_{11} = A_{\underbrace{11\ldots 11}_{2q}},\quad c_{12} = A_{\underbrace{11\ldots 12}_{2q}}, \ldots, \sum\nolimits_k \sum\nolimits_P (2 - \delta_{j_1 j_2}) c_{j_1 j_2} = A_{i_1 \ldots i_s}, \ldots, c_{NN} = A_{nn} \tag{1.6}$$

where $\sum_k$ denotes summation over all partitions of multi-indices $i_1 \ldots i_s$ into subindices $i_1 \ldots i_k$ and $i_{k+1} \ldots i_s$; $\sum_P$ denotes summation over all permutations of $i_1, \ldots, i_s$ that preserve the conditions $i_1 \leqslant \cdots \leqslant i_k$, $i_{k+1} \leqslant \cdots \leqslant i_s$, $\delta_{j_1 j_2}$ is the Kronecker delta. System (1.6) may be used to determine $c_{j_1 j_2}$ given $A_{i_1 \ldots i_s}$ and conversely, to compute $A_{i_1 \ldots i_s}$, when $c_{j_1 j_2}$ are known.

Lemma 1.2. A function $F_r(\mathbf{x})$ (1.2) defined in R_x^n is mapped by Φ (1.3) into a linear form

$$f_r(\mathbf{y}) = \sum_{j=1}^{N} b_{rj} y_j, \quad r = 1, \ldots, m \tag{1.7}$$

defined in the space R_y^N.

Proof. The highest order of terms in the sums of forms (1.2), which equals the number p, is not greater than $q\ (p \leqslant q)$. Therefore, apart from coefficients, each term in the functions $F_r(\mathbf{x})$ is equal to some unique coordinate y_j in the mapping (1.3). Consequently, the sums of forms $F_r(\mathbf{x})$ are always mapped by Φ into a linear form (1.7).

Equating the linear form $f_r(\mathbf{y})$ (1.7) to the function $F_r(\mathbf{x})$,

$$\sum_{j=1}^{N} b_{rj}y_j = \sum_{s=1}^{p} X_r^{(s)}(x, B_{rj_1\ldots i_s}), \quad r=1,\ldots,m \tag{1.8}$$

and substituting the values of $y_1, \ldots, y_N$ from (1.3), we obtain

$$b_{r1} = B_{r11\ldots11}, \quad b_{r2} = B_{r11\ldots12}, \ldots, \quad B_{rj} = B_{ri_1\ldots i_s}, \ldots, \quad b_{rN} = B_{rn} \tag{1.9}$$

Theorem 1.1. A q.f. $f(\mathbf{y})$ (1.4) is positive- or negative-definite on linear manifolds M_r

$$f_r(\mathbf{y}) = \sum_{j=1}^{N} b_{rj}y_j = 0, \quad r=1,\ldots,m; \quad m<n \tag{1.10}$$

if and only if there exist real numbers a_{ij}, λ, satisfying the recurrence relation

$$a_{ij} = \frac{1}{a_{ii}}\Big(c_{ij} + \lambda^2 \sum_{r=1}^{m} b_{ri}b_{rj} - \sum_{k=1}^{i=1} a_{ki}a_{kj}\Big) \tag{1.11}$$
$$i = 1,\ 2,\ \ldots,\ N;\ j = i,\ i+1,\ \ldots,\ N;\ i \geqslant k \geqslant 1,\ m < N$$

and the inequalities

$$a_{ii} \neq 0,\ \forall i = 1,\ 2,\ \ldots,\ N \tag{1.12}$$

Proof. Necessity. Assume that $f(\mathbf{y})$ is positive-definite for the values of $y_1, \ldots, y_N$ satisfying (1.10). Then, by Finsler's theorem /2/, there exists a number λ such that the q.f.

$$P(\mathbf{y}) = f(\mathbf{y}) + \lambda^2 \sum_{r=1}^{m} f_r^2(\mathbf{y}) \tag{1.13}$$

is positive-definite.

In /3/ we establish a test for a q.f. to be of fixed sign, which we now apply to $P(\mathbf{y})$. Equate $P(\mathbf{y})$ to a positive-definite q.f. with undetermined coefficients

$$f(\mathbf{y}) + \lambda^2 \sum_{r=1}^{m} f_r(\mathbf{y}) = \sum_{i=1}^{N} \Big(\sum_{j=i}^{N} a_{ij}y_j\Big)^2 \tag{1.14}$$

Equating coefficients of like terms on the left and right of (1.14), we find real numbers a_{ij} satisfying the recurrence relations (1.14) and inequalities (1.12).

Sufficiency. Suppose there exist real numbers λ, a_{ij} satisfying formulae (1.11) and (1.12). Then, by the above-mentioned test of /3/, the q.f. $P(\mathbf{y})$ of (1.13) is positive-definite. Hence it follows that the q.f. $f(\mathbf{y})$ is positive-definite on the linear manifolds M_r (1.10) where $P(\mathbf{y}) = f(\mathbf{y})$.

Theorem 1.2. A sufficient condition for a sum of forms $F(\mathbf{x})$ to be positive-definite on a variety M (1.2) is that there exist real numbers λ, a_{ij} satisfying the recurrence relations (1.11) and inequalities (1.12) in which the numbers c_{ij}, b_{ri} satisfy Eqs.(1.6) and (1.9), respectively.

Proof. Construct a new function from the functions $F(\mathbf{x})$ (1.1) and $F_r(\mathbf{x})$ (1.2), as follows (λ is a real number):

$$Q(\mathbf{x}) = F(\mathbf{x}) + \lambda^2 \sum_{r=1}^{m'} F_r^2(\mathbf{x}) \tag{1.15}$$

Apply the mapping Φ of (1.3) to $Q(\mathbf{x})$. When this is done, the functions $F(\mathbf{x})$ and $F_r(\mathbf{x})$ are carried into a q.f. $f(y)$ (1.4) and a linear form $f_r(\mathbf{y})$ (1.7), respectively (see Lemmas 1.1 and 1.2), so that $Q(\mathbf{x})$ itself is carried into a q.f. $P(\mathbf{y})$ of type (1.13).

Suppose that the condition of the theorem is satisfied, i.e., the coefficients of the functions $F(\mathbf{x})$ and $F_r(\mathbf{x})$ determine real numbers c_{ij}, b_{ri} such that there exist λ, a_{ij} satisfying formulae (1.11) and (1.12). Then, by Theorem 1.1, the q.f. $f(\mathbf{y})$ of (1.4) is positive-definite on the linear varieties M_r (1.10), and so the q.f. $P(\mathbf{y})$ is also positive-definite. It was proved in /4/ that under these conditions the function $Q(\mathbf{x})$ of (1.15) is also positive-definite. Since the identity $Q(\mathbf{x}) \equiv F(\mathbf{x})$ holds on M (1.2), it follows that the sum of forms $F(\mathbf{x})$ is positive definite on M.

2. The main theorems of the Lyapunov function method, concerning stability of motion,

carry over easily to the case of motion on varieties. We will first present the necessary definitions and then the stability theorems.

Suppose that the equations of perturbed motion are

$$\mathbf{x}^{\cdot} \equiv d\mathbf{x}/dt = \mathbf{X}(t, \mathbf{x}), \quad \mathbf{X}(t, 0) \equiv 0 \tag{2.1}$$

Here $\mathbf{x} = (x_1, \ldots, x_n) \in R^n$; $\mathbf{X}(t, \mathbf{x}) = (X_1(t, \mathbf{x}), \ldots, X_n(t, \mathbf{x}))$ is a vector function defined, continuous and Lipschitzian with respect to $\mathbf{x}$ in a domain

$$G(t, \mathbf{x}) = \{t, \mathbf{x} : t \in [t_0, \infty), \|\mathbf{x}\| = \sqrt{\sum_{i=1}^{n} x_i^2} \leqslant H > 0, H = \text{const}\} \tag{2.2}$$

It is known that the family of integral curves of Eqs.(2.1) belongs to the variety M defined by the equalities

$$F_r(t, \mathbf{x}) = 0, \quad r = 1, 2, \ldots, m, \quad F_r(t, 0) \equiv 0 \tag{2.3}$$

where $F_r(t, \mathbf{x})$ are continuous and continuously differentiable with respect to $\mathbf{x}$ and t, and their total derivative with respect to t along trajectories of system (2.1) vanishes.

Let us assume that the solution $\mathbf{x} = \mathbf{x}(t; t_0, x_0)$ of system (2.1) with constraints (2.3) and intiial conditions $\mathbf{x}(t_0; t_0, \mathbf{x}_0) = \mathbf{x}_0$ is defined for all $t \geqslant 0$ for which $\|\mathbf{x}\| \leqslant H$. The solution $\mathbf{x} \equiv 0$ corresponds to unperturbed motion of system (2.1) and is a member of M (2.3).

Consider the real functions $V(t, x)$ $(V(t, 0) \equiv 0)$ defined, continuous and having continuous partial derivatives $\partial V/\partial t, \partial V/\partial x_i$ $(i = 1, 2, \ldots, n)$ in the domain G (2.2), as well as their total derivatives $V^{\cdot}(t, \mathbf{x})$ with respect to time along trajectories of system (2.1),

$$V^{\cdot}(t, \mathbf{x}) \equiv \frac{dV}{dt} = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V(t, \mathbf{x})}{\partial x_i} X_i(t, \mathbf{x}) \tag{2.4}$$

Definition 2.1. A function $V(t, \mathbf{x})$ is said to be semidefinite on a manifold M (2.3) if its non-zero values on M are all of the same sign.

Definition 2.2. A function $W(\mathbf{x})$ not explicitly dependent on t is said to be positive-definite on a manifold M (2.3) if it is non-negative at every point of M and vanishes if and only if $\mathbf{x} = 0$.

Definition 2.3. A function $V(t, \mathbf{x})$ is said to be positive-definite on a manifold M (2.3) if there exists a function $W(\mathbf{x})$, not explicitly dependent on t, which is positive-definite on M and

$$V(t, \mathbf{x}) \geqslant W(\mathbf{x}) \tag{2.5}$$

Definition 2.4. A function $V(t, \mathbf{x})$ is said to have an infinitesimal upper limit on a manifold (2.3) if, for any $\varepsilon > 0$, one can find $\delta > 0$ such that the following condition holds on M:

$$|V(t, \mathbf{x})| < \varepsilon, \quad \text{if} \quad \|\mathbf{x}\| \leqslant \delta, \quad t \geqslant 0$$

Definition 2.5. A function $V(t, \mathbf{x})$ is said to have an infinitely large lower limit on a manifold M (2.3) if, for any number $A > 0$, there exists a number $B > 0$ such that the following condition holds on M:

$$|V(t, \mathbf{x})| > A, \quad \text{if} \quad \|\mathbf{x}\| \geqslant B, \quad t \geqslant 0$$

The following proposition, due to Lyapunov, was originally stated as a corollary to his stability theorem /1/:

Theorem 2.1. If there exists a function $V(t, \mathbf{x})$ which is positive-definite on a manifold M (2.3), and whose derivative $V^{\cdot}(t, \mathbf{x})$ along trajectories of system (2.1) is negative-semidefinite on M or $V^{\cdot} \equiv 0$ on M, then the unperturbed motion is stable on M.

Theorem 2.2. If there exists a function $V(t, \mathbf{x})$ which is positive-definite and has an infinitesimal upper limit on M (2.3), and moreover its derivative $V^{\cdot}(t, \mathbf{x})$ is negative-definite on M, then the unperturbed motion is asymptotically stable on M.

Theorem 2.3. If the assumptions of Theorem 2.2 hold on the manifold M (2.3) and the function $V(t, \mathbf{x})$ also has an infinitely large lower limit on M, then the unperturbed motion is asymptotically stable in the large on M.

This statement, for a domain G, is due to Krasovskii /5/.

Other stability theorems carry over to the case under consideration. The proofs are practically the same, expect that one should bear in mind that the motion of the representative point of the system takes place on a manifold M (2.3).

3. We will now derive the sufficient conditions for the motion of polynomial systems on a manifold M (1.2) to be stable. Suppose that the perturbed motions of the system are described by equations of the form

$$x_\beta^{\cdot} = \sum_{s=1}^{2h-1} X_\beta^{(s)}(\mathbf{x}, C_{\beta i_1 \ldots i_s}), \quad \beta = 1, 2, \ldots, n \tag{3.1}$$

where $C_{\beta i_1 \ldots i_s}$ are real numbers, h, m integers, the coordinates $x_1, \ldots, x_n$ satisfy Eqs.(1.2) and the following conditions hold:

$$\frac{dF_r}{dt} = \sum_{\beta=1}^{n} \frac{\partial F_r}{\partial x_\beta} x_\beta^{\cdot} \equiv 0 \tag{3.2}$$

Throughout the sequel these conditions will be assumed to be valid.

We shall find conditions linking the coefficients $C_{\beta i_1 \ldots i_s}$ and $B_{r i_1 \ldots i_s}$ (1.2), under which the unperturbed motion on the manifold M (1.2) is asymptotically stable in the large. This will be done using Theorem 2.2.

The Lyapunov function will be sought in the set of negative-definite functions

$$V(\mathbf{x}) = -\frac{1}{2} \sum_{\alpha=1}^{N} \Big(\sum_{r=1}^{k} X_\alpha^{(r)}(\mathbf{x}, d_{\alpha i_1 \ldots i_r}) \Big)^2 \tag{3.3}$$

where $X_\alpha^{(r)}(\mathbf{x}, d_{\alpha i_1 \ldots i_r})$ is a multilinear form of degree r with real constant coefficients $d_{\alpha i_1 \ldots i_r}$ $(\alpha = 1, 2, \ldots, N;\ i_1, \ldots, i_r = 1, \ldots, n)$, forming a non-singular $(N \times N)$ matrix, e.g., an upper triangular matrix with non-zero diagonal elements, and k is an integer.

The total derivative of the function $V(x)$ (3.3) with respect to t along trajectories of system (3.1) is a sum of forms of type (1.1):

$$V^{\cdot}(\mathbf{x}) = \sum_{\beta=1}^{n} \frac{\partial V}{\partial x_\beta} x_\beta^{\cdot} = \sum_{s=2}^{2q} X^{(s)}(\mathbf{x}, A_{i_1 \ldots i_s}) \tag{3.4}$$

where $q = k + h - 1$ and the coefficients $A_{i_1 \ldots i_s}$ are determined by reducing like terms after scalar multiplication of the vector $(\partial V/\partial x_1, \ldots, \partial V/\partial x_n)$ by the vector $(x_1^{\cdot}, \ldots, x_n^{\cdot})$; they equal the sums of the appropriate products of the coefficients $d_{\alpha i_1 \ldots i_r}$ and $C_{\beta i_1 \ldots i_s}$ of (3.3) and (3.1), respectively.

Theorem 1.1 yields conditions for the derivative $V^{\cdot}$ to be positive-definite on the manifold M (1.2). Hence we arrive at the following assertion.

Theorem 3.1. A sufficient condition for the unperturbed motion of system (3.1) to be asymptotically stable in the large on the manifold M (1.2) is that there exist real numbers λ, a_{ij} satisfying the recurrence relations (1.11) and condition (1.12), where the coefficients c_{ij}, b_{ri} are determined from Eqs.(1.6) and (1.9), respectively.

We now consider a linear system. System (3.1) with $h = 1$ is a system of linear differential equations with constant coefficients:

$$x_\beta^{\cdot} = \sum_{i=1}^{n} C_{\beta i} x_i, \quad \beta = 1, 2, \ldots, n \tag{3.5}$$

Eqs.(1.2) with $p = 1$ determine linear manifolds M_r:

$$\sum_{i=1}^{n} B_{ri} x_i = 0, \quad r = 1, 2, \ldots, m; \quad m < n \tag{3.6}$$

The function (3.3) with $k = 1$, $N = n$, is a negative-definite q.f.:

$$V(\mathbf{x}) = -\frac{1}{2} \sum_{\alpha=1}^{n} \Big(\sum_{i=1}^{n} d_{\alpha i} x_i \Big)^2 \tag{3.7}$$

Theorem 3.2. The conditions of Theorem 3.1 with $N = n$ are necessary and sufficient for the linear system (3.5) to be asymptotically stable on linear manifolds (3.6).

Proof. Necessity. Suppose that the linear system (3.5) is asymptotically stable under conditions (3.6). It follows from Lyapunov's theorem - the existence of a Lyapunov q.f. for asymptotically stable linear systems - that the total derivative of the function (3.7) along trajectories of system (3.5) is positive-definite. Then the q.f.

$$P(\mathbf{x}) = V^{\cdot} + \lambda^2 \sum_{r=1}^{m} \Big(\sum_{i=1}^{n} B_{ri}x_i\Big)^2 \tag{3.8}$$

is positive-definite on the linear manifolds (3.6).

A criterion for this q.f. to be positive- or negative-definite is provided by formula (1.11) and inequalities (1.12).

Sufficiency. Suppose there exist numbers λ, a_{ij} is indicated in the theorem, satisfying (1.11) and (1.12). Then /4/ the q.f. $P(\mathbf{x})$ is positive-definite. Hence it follows that $P(\mathbf{x}) = V^{\cdot}$, the total derivative of the q.f. $V(\mathbf{x})$ (3.7) along trajectories of system (3.5) is positive-definite on the linear manifolds (3.6). By Theorem 2.2, this implies that system (3.5) is asymptotically stable on the linear manifolds (3.6).

Example. We shall show that the solution $\mathbf{x} \equiv 0$ of the system

$$\begin{aligned} x_1^{\cdot} &= -2x_1x_2 + 2x_2^2 - 2x_2^3 - 2x_1x_2^2 \\ x_2^{\cdot} &= x_1 - x_2 + x_2^2 + x_1x_2 \end{aligned} \tag{3.9}$$

is asymptotically stable on the manifold M

$$F_1(x_1, x_2) = x_1 + x_2^2 = 0 \tag{3.10}$$

Since the total derivative of $F_1(x_1, x_2)$ with respect to time is zero, the integral curves of system (3.8) lie on the manifold M.

To solve the problem, we make use of Theorem 3.1. Consider the Lyapunov function

$$V = -{}^1/_2\,[(d_{11}x_1 + d_{12}x_2)^2 + (d_{22}x_2)^2]$$

where d_{11}, d_{12}, d_{22} are arbitrary real numbers.

We evaluate the total derivative $V^{\cdot}$ of this function with respect to t along trajectories of system (3.8) and construct the function

$$\begin{aligned} Q(\mathbf{x}) = V^{\cdot} + \lambda^2F_1 = 2d_{11}^2x_1^2x_2^2 + 2(d_{11}^2 + d_{12}d_{11})\,x_1x_2^3 + (2d_{11}^2 - \\ d_{11}d_{12})\,x_1^2x_2 + (-2d_{11}^2 + d_{11}d_{12} - d_{12}^2 - d_{22}^2)\,x_1x_2^2 + 2d_{11}d_{12}x_2^4 - \\ (2d_{11}d_{12} + d_{12}^2 + d_{22}^2)\,x_2^3 - d_{11}d_{12}x_1^2 + (d_{11}d_{12} - d_{12}^2 - d_{22}^2)\,x_1x_2 + \\ (d_{12}^2 + d_{22}^2)\,x_2^2 + \lambda^2(x_1 + x_2)^2 \end{aligned} \tag{3.11}$$

A mapping taking $Q(\mathbf{x})$ into a q.f. is

$$y_1 = x_1x_2, \quad y_2 = x_2^2, \quad y_3 = x_1, \quad y_4 = x_2 \tag{3.12}$$

This mapping takes $Q(\mathbf{x})$ into the q.f.

$$P(\mathbf{y}) = f(y) + \lambda^2 f_1(\mathbf{y}) = \sum_{j_1, j_2}^{4} c_{j_1j_2}y_{j_1}y_{j_2} + \lambda^2(b_{12}y_2 + b_{13}y_3)^2, \quad c_{j_1j_2} = c \tag{3.13}$$

Equating the functions $Q(\mathbf{x})$ and $P(\mathbf{y})$, substituting the values of y_1, y_2, y_3, y_4 from (3.12) and comparing coefficients of like terms, we obtain

$$\begin{aligned} &c_{11} = 2d_{11}^2, \quad c_{12} = c_{21} = d_{11} + d_{12}d_{11}, \quad c_{13} = c_{31} = d_{11}^2 - {}^1/_2 d_{11}d_{12} \\ &c_{14} = c_{41} = {}^1/_2(-2d_{11}^2 + d_{11}d_{12} - d_{12}^2 - d_{22}^2), \quad c_{22} = 2d_{11}d_{12} \\ &c_{23} = c_{32} = 0, \quad c_{33} = -d_{11}d_{12}, \quad c_{34} = c_{43} = {}^1/_2(d_{11}d_{12} - d_{12}^2 - d_{22}^2) \\ &c_{44} = d_{12}^2 + d_{22}^2, \quad b_{12} = 1, \quad b_{13} = 1 \end{aligned}$$

Now, using the recurrence relations (1.11), we compute the numbers a_{ij} and check the validity of inequalities (1.12). The result is

$$\begin{aligned} &a_{11} = a_{13} = 1, \quad a_{12} = a_{24} = -{}^1/_2, \quad a_{14} = {}^1/_2, \quad a_{23} = {}^3/_2 \\ &a_{22} = a_{33} = 2, \quad a_{34} = -{}^5/_4, \quad a_{44} = ({}^1/_4)\sqrt{39}, \quad \lambda = {}^5/_2. \end{aligned}$$

Thus the conditions of Theorem 3.1 are satisfied. The solution $\mathbf{x} \equiv 0$ is therefore asymptotically stable in the large on the manifold (3.10).

REFERENCES

1. LYAPUNOV A.M., The general problem of stability of motion. In: Collected Works, 2, Moscow, Izd. Akad. Nauk SSSR, 1956.
2. BELLMAN R., Introduction to Matrix Theory /Russian translation/, Moscow, Nauka, 1969.
3. AMINOV A.B. and SIRAZETDINOV T.K., Conditions for even forms to be of fixed sign and for stability in the large of non-linear homogeneous systems. Prikl. Mat. Mekh., 48, 3, 1984.
4. AMINOV A.B. and SIRAZETDINOV T.K., Lyapunov functions for investigating the stability in the large of non-linear systems. Prikl. Mat. Mekh., 49, 6, 1985.
5. KRASOVSKII N.N., Some Problems in the Theory of the Stability of Motion, Moscow, Fizmatgiz, 1959.

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